Square Root Approximations in Old Babylonian Mathematics: YBC 7289 in Context

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We consider several aspects of the role and evaluation of the four-sexagesimal-place approximation to $\sqrt{2}$ on the well-known Old Babylonian tablet YBC 7289. By referring to what is known about OB school texts, we show that this text is most probably a school exercise by a trainee scribe who got the approximation from a coefficient list. These coefficient lists are briefly described, with their use in geometrical problems. We consider other texts involving square roots and derive an algorithm for evaluating them, which complies with all known OB examples, from a simple geometrical construction of the type that seems to underlie many other OB procedures.   © 1998 Academic Press

Nous considérons plusieurs aspects du rôle et de l'évaluation de l'approximation à quatre places sexagésimales de $\sqrt{2}$ sur cette tablette paléo-babylonienne. En nous référant à l'état des connaissances quant aux textes scolaires paléo-babyloniens, nous montrons comment ce texte est très probablement un exercice scolaire exécuté par un apprenti scribe qui a obtenu l'approximation à partir d'une liste de coéfficients. Nous décrivons en bref ces listes de coéfficients, ainsi que leur utilisation dans les problèmes de géométrie. Nous considérons d'autres textes traitant de racines carrées , et pour les évaluer nous dérivons un algorithme qui est en accord avec tous les exemples paléo-babyloniens connus; ceci à partir d'une construction géométrique simple du type qui semble être caractéristique de beaucoup d'autres procédures paléo-babyloniennes.   © 1998 Academic Press

YBC 7289, from the Yale Babylonian Collection, is one of the best-known Old Babylonian mathematical clay tablets.[1] Its exact provenance and dating are unknown, but the round shape of the tablet and the palæography suggest that it was written by a trainee scribe somewhere in southern Mesopotamia (modern Iraq) some time in the first third of the second millennium BC. It was first published in [19, 43], and has since appeared, with a drawing or photograph, in most accounts of Babylonian mathematics; see Fig. 1. Here, for example, is part of its description in [20, 35]:

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[1] The following abbreviations are used in this article: (BM) British Museum, Department of Western Asiatic Antiquities, London; (IM) Iraq Museum, Baghdad; (*MCT*) *Mathematical Cuneiform Texts* [=19]; (OB) Old Babylonian period (c. 2000–1600 BC); (*TMS*) *Textes mathématiques de Suse* [=3]; (VAT) Vorderasiatische Abteilung, Tontafeln, Staatliche Museen, Berlin; (YBC) Yale Babylonian Collection, New Haven.
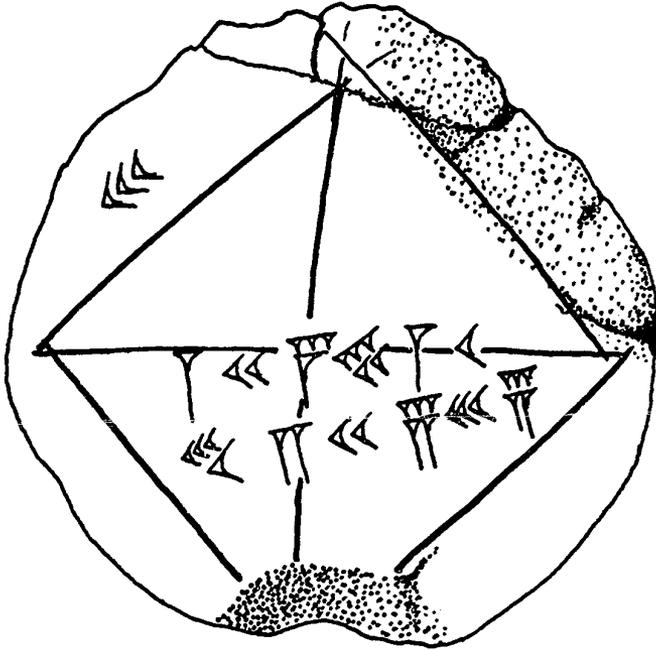
FIG. 1. The Old Babylonian tablet YBC 7289. (From Asger Aaboe, *Episodes from the Early History of Mathematics*, Washington, DC: The Mathematical Association of America, 1964. Reprinted by permission of the Mathematical Association of America.)

On it is drawn a square with its two diagonals. The side shows the number 30, the diagonals the numbers 1, 24, 51, 10 and 42, 25, 35. The meaning of these numbers becomes clear if we multiply 1, 24, 51, 10 by 30, an operation which can be easily performed by dividing 1, 24, 41, 10 by 2 because 2 and 30 are reciprocals of each other. The result is 42, 25, 35. Thus we have obtained from $a = 30$ the diagonal $d = 42;25, 35$ by using $\sqrt{2} = 1;24, 51, 10$.

(Note that the "sexagesimal semicolons" indicating the absolute size of the numbers are only inserted in the last sentence. There was no way of indicating the absolute value of a number in cuneiform, either through final zeros or through a sexagesimal semicolon. When necessary, scribes made reference to metrological units, or wrote a word such as "sixty" or "thousand" after a number to indicate its approximate magnitude.)

Conventions for transliterating numbers have become simpler and clearer in the last 50 years, and ours follow the now-standard procedures laid down by Friberg [7, 534]: numbers are transliterated with no indication of absolute value; spaces are left between the sexagesimal places; missing tens and/or units are represented by zeros; and sexagesimal semicolons are added in translation and commentary. Also, following standard Assyriological practice, Akkadian is transliterated in *italic* type-

face; logograms of Sumerian origin are written in SMALL capitals; and cuneiform signs whose contextualized reading is uncertain are given in LARGE capitals.

When one of us (DHF) used to discuss this tablet with students, he was for some years compelled to make the following remark which he felt must surely be known and obvious to the experts, but which seems only to be found in one early discussion of the text, quoted in full below: the interpretation of 30 as 0;30 = 1/2 and hence 42 25 35 as 0;42 25 35 is surely less arbitrary, for the numbers written against the diagonal then represent approximations to $\sqrt{2}$ and $1/\sqrt{2}$. Thus we have a reciprocal pair of numbers with a geometric interpretation of them, and all of the numbers on the tablet are closely related to 2. The importance of reciprocals in Babylonian mathematics is well known. The reciprocal of n was written with the logograms IGI-n-GÁL.BI, or IGI-n,[2] and many tables of reciprocals have been found: lots containing a standard set of reciprocals of regular numbers (see most recently [18, 245–253], with references to earlier literature), some containing approximations to irregular numbers, a remarkable later Seleucid[3] table of six-place regular reciprocals by Ina-qibīt-Anu ("At the command of the Sky God")[4] and a few fragments of other such Seleucid extended tables [1; 8]. So this interpretation relates YBC 7289 to a well-attested preoccupation of Babylonian mathematics.

The first reference we have seen to such an interpretation is in the invaluable [6, 64]. In his dense three-typescript-page summary of MCT [=19], Friberg describes YBC 7289 as "a lenticular school tablet with a geometric drawing displaying the very good approximations $\sqrt{2} \approx 1.24\ 51\ 10$ [and $1/\sqrt{2} \approx [.]42\ 25\ 35$]."

We could be content with thinking of 1 24 51 10 and 42 25 35 as mutual approximate reciprocals if this tablet could not be seen in its context. But the second author, ER, one of DHF's ex-students who now works on Mesopotamian mathematics, can fill in a lot more detail that puts this simple and attractive interpretation into question, just as it may be passing into general circulation.[5]

From the mid-third millennium onwards, trainee scribes typically did some types of school work on more-or-less round or square tablets between 8 and 12 cm in diameter. These erasable tablets were known as "hand tablets" (Sumerian IM.ŠU, Akkadian *imšukkum*), as they fitted very comfortably into the palm of the hand

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[2] The signs GÁL and BI are Sumerian for "is" and "its" respectively; IGI is the logogram for Akkadian *pānum* "reciprocal" (literally "face" or "eye"). The phrase *m* IGI-n-GÁL.BI most usually occurs in tables of reciprocals: "*m*: n is its reciprocal". IGI-n is more commonly found in problem texts—although one occasionally also sees the Akkadianised *igûm*.

[3] I.e., dating to the period 311–126 BC, some 1500 years after the Old Babylonian period. It is named after Seleucus I Nicator, who (with Ptolemy I Soter and Antigonus II Gonatas) was one of Alexander's three generals who split and ruled the three parts of his empire after his death.

[4] In his interesting article [16, with later corrections], Knuth describes this table and pays homage to its author: "Thus Inakibit [sic] seems to have the distinction of being the first man in history to solve a computational problem that takes longer than one second of time on a modern electronic computer!" But this article *must* be collated against its correction.

[5] Neugebauer and Sachs [19, 39] did not offer this explanation of the tablet although they must surely have considered it.

[4, 75; 11, 70–71].[6] They appear to have been used for rough work of various sorts, from handwriting practice and memorization of short texts to mathematical work such as that on YBC 7289. The calculations on IM.ŠU were probably carried out while practising how to write business documents such as accounts and land surveys, or while solving mathematical problems. There are also no fewer than 20 compilations of mathematical problems on long, rectangular tablets which end with colophons giving the (sometimes inaccurate!) number *n* of problems on them as *n* IM.ŠU [17, 176–177]. Without exception these compilations consist of the statements of problems and sometimes their numerical answers, but they do not give the solutions. In other words, they must be collections of exercises to be done on IM.ŠU tablets.[7]

The fact that the handwriting on YBC 7289 is unusually large (c. 8 mm in height) also suggests that the tablet was written by a learner: the scribes' handwriting tended to get smaller with practice.

So we have established with some confidence the function of YBC 7289: it was rough work written by a student while solving a school problem. But what might that problem have been? To answer this question we now introduce the cuneiform tablet BM 15285 [9; 22, Sect. 12.1.2; 25], an OB illustrated geometrical "textbook" now containing 30 or so of an original 40 problems on finding the areas of certain figures inscribed in squares. In each case the problem is set, but no method of solution or numerical answer given. Problems (vii)–(xii) all concern shapes based on squares inscribed obliquely in the outer square—i.e., squares with sides whose lengths are $1/\sqrt{2}$ of the outer sides. In each case the outer square is described as having length 1 UŠ.[8] To perform the calculations required, the student would have first had to express this length in terms of the standard unit, the rod, so here 1 UŠ = 60 rods = 1 in sexagesimal, again. Although it is highly improbable (but admittedly just possible) that the scribe of YBC 7289 was setting out to solve a problem from BM 15285 itself, we now have found a plausible context for allowing us to interpret the numbers on it. In this scenario, the length can be read as 30;00

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[6] *Imšukkum* was also the word for a chamber pot: this pun was no doubt an endless source of hilarity, as, for example, in the first millennium B.C. school tale of Ninurta-pāqidāt's dog-bite, which climaxes as the unfortunate Ninurta-pāqidāt is chased from the scribal city of Nippur by schoolchildren hurling *imšukkū* after him [11].

[7] This hypothesis is strengthened by the tablets YBC 4657, YBC 4663, and YBC 4662 [19, texts G, H, and J]. The first of these sets contains 31 problems on KI.LÁ excavations, described at the end as 31 IM.ŠU KI.LÁ "31 exercises on excavations;" the other two give the step-by-step solutions to problems (i)–(viii) and (xix)–(xxviii) of YBC 4657; and neither contains a colophon of any sort, although one might expect an identifying phrase similar to that on the first tablet. (For instance, solutions of this type are called *kibsātum* "steps" (derived from the Akkadian verb *kabāsum* "to step on," "to tread") in colophons of tablets from Sippar; see [17, 184] for a full list of occurrences.) Both of these two tablets are long and rectangular, measuring roughly 75 by 190 mm [19, pl. 32–33], and so cannot be IM.ŠU in the sense of round tablets. If neither the objects YBC 4663 and YBC 4662 nor the text on them are IM.ŠU, then the word must refer solely to the contents of YBC 4657, namely, the problems to be solved but not the pertinent procedures.

[8] The Old Babylonian metrological units referred to in this article are [21, 459, 479]: Length measure: 1 cubit (KÙŠ) = 30 fingers ≈ 0.5 m; 1 rod (NINDAN) = 12 cubits ≈ 6 m; 1 UŠ = 60 rods ≈ 360 m; 1 stage (DANNA or *bērum*) = 30 UŠ ≈ 10.8 km. Area measure: 1 plot (SAR) = 1 rod × 1 rod ≈ 36 m².

(rods), and the number below the diagonal as 42;25 35 (rods). The number above it would still be 1;24 51 10—the coefficient of the diagonal of a square.

This approximation to $\sqrt{2}$ would have been copied by the scribe from a so-called coefficient list—a list of numbers useful for doing calculations. These reference lists were a vital part of Mesopotamian mathematics; Neugebauer and Sachs describe them as ''lists of numbers (to which are added short explanations) which occur in the mathematical texts. We find here, e.g., coefficients referring to bricks, work assignment, etc.—in short, just those parameters which must be known by anyone dealing with various types of mathematical texts.'' [19, 38]

Eight such Old Babylonian lists are known, and a total of 49 different geometrical coefficients attested in lists or problems [22, Sect. 3]. One of these lists, YBC 7243 [19, text Ue], records in line 10:

1 24 51 10 ṣi-li-ip-tum ⟨ša⟩ íB.SI₈   1;24 51 10, the diagonal of a square.

Every Old Babylonian geometrical shape has a ''defining component,'' usually an outer edge. Often these defining components have the same Akkadian name as the shape as a whole; for instance, a square and the side of a square are both called *mithartum* (written logographically in YBC 7243) [cf. 13; 5]. So when an entry in a coefficient list says

0;05, a circle

[e.g., 19, text Ud: line 20, text Ue: line 1] this means that the area of a circle can be calculated by multiplying the square of the circumference by the coefficient 0;05 (both circumference and circle are called *kippatum*). All geometrical coefficients which are just called after the shape itself relate the (square of the) defining component to the area. Similarly, coefficients named after some other part of a figure also define the length or area of that part in terms of that figure's defining component [22, Sect. 3]. So the coefficient list entry

1;24 51 10, the diagonal of a square

means that the diagonal of *any* square is 1;24 51 10 times longer than its defining component, namely, its side.

But, although we can now say that the author of YBC 7289 most probably took the value of $\sqrt{2}$ from a reference list, this still begs the question of how such an accurate value was determined in the first place. So we now need to look into the approximation of square roots.

As was said above, the same name can refer to the side of a square or the square itself so, to help the reader, we shall use lower-case names such as ''approx,'' ''new approx'' for lengths, and capitalized names such as ''Number'' and ''Bit'' for areas.[9]

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[9] These short or abbreviated names are a loose allusion to the coss tradition of mediaeval algebra, and our use here of their description as ''names'' emphasizes further that we are describing presymbolic arithmetical manipulations. Similarly, we use ''Square of approx'' (the acceptable alternative ''Square on approx'' would have echoes of the geometrical basis of OB arithmetical procedures, which we shall here go on to exploit and which has been convincingly demonstrated by Høyrup, on the basis of a close study of OB vocabulary; see [14]) and ''side of Area'' (an abbreviation for ''the length of the sides of a square equal to the given area''). However, we have stopped short of the final step of replacing all
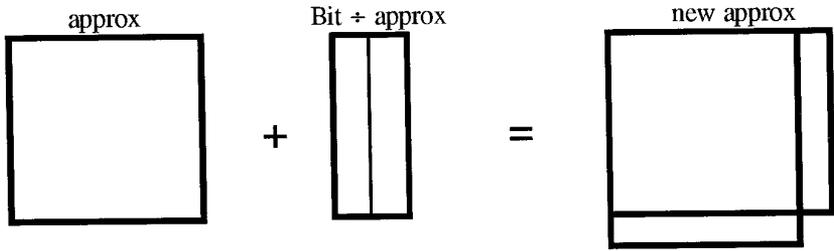
approx   Bit ÷ approx   new approx

FIGURE 2

So suppose we want to evaluate the "side of a Number" (our square root). We start from some approximation, and let us first examine the case where this is an underestimate, so

$$\text{Number} = \text{Square of approx} + \text{Bit}$$

which, geometrically, can be represented by the sum of a square with sides approx and the leftover Bit. Now express this Bit as a rectangle with sides approx and, therefore, Bit ÷ approx, or, OB style, Bit × IGI approx; cut this in two lengthwise, and put the halves on two adjacent sides of the Square of approx, as shown in Fig. 2. Hence

$$\text{new approx} = \text{approx} + \text{half of Bit} \times \text{IGI approx,}$$

and it will clearly be an overestimate because of the bite out of the corner. Those who still feel the need for an algebraic description of the process need only retain the initial letters of our description:

$$\sqrt{N} = \sqrt{(a^2 + B)} \approx a + \frac{1}{2}\frac{B}{a}.$$

Before going any further, here is an OB illustration of the use of such an approximation, in the translation of problem (xviii) from the cuneiform tablet BM 96957 + VAT 6598 (see [22, Sect. 12.1.4; 24], with references to earlier literature):

A gate, of height 1/2 ⟨rod⟩ 2 cubits, and breadth 2 cubits. What is its diagonal? You: square 0;10, the breadth. You will see 0;01 40, the base. Take the reciprocal of 0;40 (cubits), the height;

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of the arithmetical symbols +, −, × and ÷ (this last not corresponding to an OB operation) by words, so as not to make the descriptions too clumsy. We had hoped to use adaptations of the Old Babylonian dialect of Akkadian for describing and manipulating these quantities, but this would have been too unwieldy and obscure; for example, "Number is as much as the sum of the square of the approximation and the Bit" and "the new approximation is as much as the sum of the approximation and the Bit broken in half by the reciprocal of the approximation multiplied."
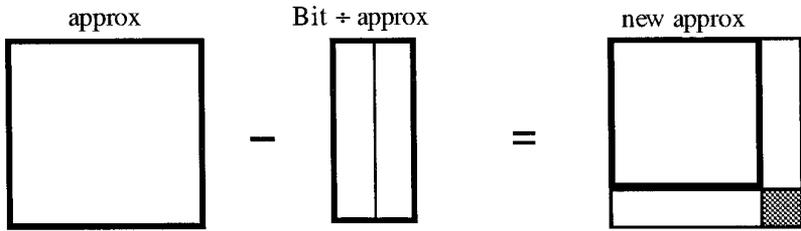
FIGURE 3

multiply by 0;01 40, the base. You will see 0;02 30. Break in half 0;02 30. You will see 0;01 15.
Add 0;01 15 to 0;40, the height. You will see 0;41 15. The diagonal is 0;41 15. The method.

As often is the case, the problem is stated in everyday units, here rods and cubits; these are then converted into sexagesimal fractions of a rod and manipulated as sexagesimal fractions throughout, but then the scribe forgets to translate the sexagesimal answer back into rods and cubits at the end. As to the method, the scribe evaluates the diagonal, the side of the sum of the squares on the height and the breadth, as the height plus half of the square of the breadth times IGI height, and note also that the breadth is rather smaller than the height, clearly a condition for the new approximation to be reasonably accurate.

Now consider what happens if our first approximation is too large, so that

$$\text{Number} = \text{Square of approx} - \text{Bit}.$$

Here, we subtract the two halves of the rectangles Bit × IGI approx from the Square of approx (Fig. 3), so that the square in the corner where the two rectangles overlap is again to be subtracted from the new approximating square with sides

$$\text{new approx} = \text{approx} - \text{half of Bit} \times \text{IGI approx},$$

and this new approximation is again too large:

$$\sqrt{N} = \sqrt{(a^2 - B)} \approx a - \frac{1}{2}\frac{B}{a}.$$

And, again, an OB illustration: if an equilateral triangle[10] has side 1, its height[11] will be $\sqrt{(1 - (0;30)^2)}$. Our approximation procedure then gives

[10] Old Babylonian geometrical terminology does not distinguish between classes of triangles—all are called *santakkum*, "wedge"—but there are basically three types: (approximate) right triangles, such as those on YBC 7289, formed by halving a square or rectangle along its diagonal; symmetric triangles composed of two identical right triangles back to back; and equilateral triangles, which are special cases of the symmetric triangle [cf. 7, 554]. The idea of squareness or perpendicularity or approximate right-angledness was the nearest thing to a concept of angle in the Old Babylonian period. The modern division of the circle into 360° ultimately derives from Mesopotamian *astronomical* texts of the first millennium B.C. in which time was measured with respect to length units.

[11] The Akkadian word is *muttarrittum*, literally "thing that always goes down;" *tallum* "long transversal" is also often used in the same sense. (This last word also has many nonmathematical meanings, including "door-lintel" and "diaphragm.")

$$\sqrt{(1 - 0;15)} \approx 1 - 0;07\ 30 = 0;52\ 30,$$

the standard OB approximation for this. Hence the area of such an equilateral triangle will be approximately $\frac{1}{2} \times 0;52\ 30 = 0;26\ 15$. These values are explained most explicitly in lines 7–8 of IM 52916 rev., a coefficient list from the small administrative town of Shadupûm, very close to modern Baghdad [12]:

SAG.KAK-*kum ša sa-am-na-*[*tu na*]-*ás-ḫa* 26 15 *i-*[*gi-gu-bu-šu*]	A triangle from which an eighth part is sub-tracted:[12] 0;26 15 is its coefficient.
ta-al-li SAG.KAK-*ki* 52 30 *i-*[*gi-gu-bu-šu*]	The long transversal of a triangle: 0;52 30 is its coefficient.

This analysis is also supported by the Susa coefficient list *TMS* 3 [3, texte III]. Lines 27–28 contain coefficients for the areas of a regular hexagon and heptagon, respectively:

2 37 30 IGI.GUB *šà* SAG.6	2;37 30, the coefficient of a hexagon
3 41 IGI.GUB *šà* SAG.7	3;41, the coefficient of a heptagon.

A closely related fragment of tablet was also found at Susa, *TMS* 2 [3, texte II]. The obverse shows a hexagon composed of six equilateral triangles, with a heptagon made up of seven symmetric triangles on the reverse.[13] The coefficients are based on the assumption that the base of each figure has length 1. Then the hexagon will be composed of six equilateral triangles of length 1 and area 0;26 15, so the area of the whole hexagon is $0;26\ 15 \times 6 = 2;37\ 30$.

This is confirmed by the figure on the obverse of *TMS* 2, in which the radius and triangular segment length are both marked 30 (or some power of 60 times 30). The numeral 6 33 45 within the segment must then be its area $30^2 \times 0;26\ 15 = 6\ 33;45$. The area of the whole hexagon is not given, but must then be $6 \times 6\ 33;45 = 39\ 22;30$.

The divided heptagon on the other side of *TMS* 2 has one division line marked 35 UŠ, "35, the length," and although no markings are visible on the sides, we might guess that they were meant to be 30 units long, as in the hexagon on the obverse. So let us explore how the value 3;41 on *TMS* 3 for the area of the heptagon might have been obtained if we assume that the division lines of a heptagon of side 1 are 7/6, or 1;10. The perpendicular height of one of the triangular segments will be $\sqrt{(1;10)^2 - (0;30)^2} = \sqrt{1;06\ 40}$, so write this as $\sqrt{1 + 0;06\ 40}$ and apply the procedure we found on BM 96957 + VAT 6598: the root will be approximately 1;03 20, the area of a triangular segment $1;03\ 20 \times 0;30 = 0;31\ 40$, and the total area $7 \times 0;31\ 40 = 3;41\ 40$. Finally, since the approximation we are using is an overestimate, truncate this to the two-place value 3;41.

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[12] This interesting description suggests that the area of an equilateral triangle could also be found by subtracting an eighth from the area of a right triangle of the same side (found by multiplying the square of the side by the coefficient 0;30): $0;26\ 15 = 7/8 \times 0;30$. This is highly reminiscent of the procedure for finding the area of a heptagon in *TMS* 2; see below.

[13] It is not necessary to assume that the polygons are inscribed in circles, *pace* [3, 32; 7, 556–557]. Although the photographs of *TMS* 2 [3, pls. 2–3] show very faint traces of arcs outside some sides of the polygons, they seem to have served solely as drawing guidelines, and were subsequently erased by the scribe. Compare the photograph of *TMS* 1 [3, pl. 1], in which the circle is an essential component of the diagram and has not been erased.

In the diagram on the reverse of *TMS* 2 the length of the segment is marked 35 UŠ, while inside is written

⌈SAG⌉.7 *a-na* 4 *te-*[*ṣi*]-*ip-ma*   A heptagon. You multiply (it) by 4 and you sub-
ší-in-ší-ra-ti   tract a twelfth and (you will see) the area.
ta-na-as-sà-ḫa-ma A.ŠÀ

Although the restoration of the first sign is tentative, it is clear that these concise instructions give a further approximation of the heptagon coefficient, equivalent to 3;40 (= 4 − 0;05 × 4). The procedure loses little accuracy and is arithmetically much simpler, requiring the square of the outer side to be multiplied by two one-place regular numbers instead of by the irregular 3;41.

The reader who has followed the approximation procedure thus far will appreciate the following two points:

For the procedure to yield a good result, the Bit should be small compared with the initial Square of approx. For this reason, none of the approximations considered so far will be especially accurate.

For the normal application of the procedure, the initial approximation should be a regular number so that its reciprocal, IGI approx, is known; hence only one step of the procedure will usually be possible. But if a more accurate result is required, we shall examine below the following two procedures: either to use an approximation to its reciprocal, or to adjust this initial approximation to a nearby regular number.

With this in mind, let us try to find a good approximation to $\sqrt{2}$. We give the first step below first in fractions (for the modern reader's convenience), and then in sexagesimals,

$$\sqrt{2} = \sqrt{\left(\frac{3}{2}\right)^2 - \frac{1}{4}} \approx \frac{3}{2} - \frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{3}{2} - \frac{1}{12} = \frac{17}{12},$$

or

$$\sqrt{2} = \sqrt{(1;30^2 - 0;15)} \approx 1;30 - \tfrac{1}{2} \times 0;15 \times 0;40 = 1;30 - 0;15 = 1;25,$$

and in fact we find this very value for the diagonal of a square in line 31 of *TMS* 3, the coefficient list from late Old Babylonian Susa of the polygons discussed above [2, texte III]:

1 25 IGI.GUB *šà* BAR.TA *šà* NIGIN   1;25 the coefficient of the diagonal of a square.

Now 17/12 = 1;25 is not a regular number so, if we are to apply the procedure again, we have the two alternatives:

Either we can suppose that some approximation was known to its reciprocal, IGI 1;25 = 0;42 21 10 35 …, so that we can write

$$\sqrt{2} = \sqrt{(1;25^2 - 0;00\ 25)} \approx 1;25 - \tfrac{1}{2} \times 0;00\ 25 \times 0;42\ 21\ 10\ 35 \dots$$
$$= 1;24\ 51\ 10\ 35\ 17 \dots$$

which, since this is an overestimate, we truncate to 1;24 51 10, the value on YBC 7289. Concerning this, we restrict ourselves to the remark that only two OB tables of approximate reciprocals are known: M10 [6, 545, with references to earlier literature], which lists poor approximations to the reciprocals of 7, 11, 13, 14, and 17; and YBC 10529 [19, 16], which gives, with some errors, up to 4 places of the approximate reciprocals of numbers from 56 to 1 20 at intervals of 1.[14] To say more about the reciprocals of irregular numbers would take us too far from our topic here.

Or we can replace the nonregular 1;25 with a nearby regular approximation and repeat this procedure with that value. Since our approximation is an overestimate, it will be best to take the closest regular number less than our current approximation, unless the closest greater is very much closer. This, alas, is what tends to happen with our example here: the closest two-place sexagesimal fraction is the rather distant 1;21,[15] which yields the approximation 1;24 56 40, much too big and inaccurate a result. Then, if we go to three-place sexagesimal fractions, the closest regular number is on the wrong side of 1;25, namely 1;25 20, greater than 1;25, and this generates the new approximation 1;24 51 15, which is still not good enough, while the next smaller regular number, 1;23 20, is rather distant and gives the approximation 1;24 52.[16] The next step would be to go to four-place sexagesimal fractions. As we remarked earlier, there are some remarkable later, Seleucid, six-place reciprocal

[14] Judging by the amount missing at the top of the tablet (which seems originally to have contained 24 lines per side), the table began with 48 and its reciprocal. We might reasonably infer that it was one of a pair, the first of which listed reciprocals up to 47. Bruins [2] has given an analysis of this table which we do not find convincing, since it is based on manipulations of the geometric series $F(p) = 1 + p^2 + p^4 + \cdots$ to give IGI$(1 \pm p) = F(p) \mp pF(p)$, with estimates of the interval in which this approximation is valid. This seems to us to involve too much of the wrong kind of knowledge for the OB period. Bruins's interpretation also depends on his assumption that tables such as YBC 10529 and W-B 1923-366 are "symmetrical about 1" when in fact they are no such thing. ("W-B 1923-366" has since been recatalogued as Ash 1923.366 and is now on display in the Ashmolean Museum, Oxford; curious readers may check for themselves.) Further, Bruins misunderstands Ina-qibīt-Anu's big reciprocal table already mentioned above. Although it indeed contains only reciprocal pairs in the range 1 to 2, which he is looking for, its colophon explicitly states that it is the first tablet (*pirsu rēštu*) of a series which the tablet does not complete (NU AL.TIL). Bruins, though, wishes to understand *rēštu* not as the commonly attested adjective "first" but as the noun "head" with a transferred meaning such as "starting point." But the Akkadian for this would be *rēš pirsi*, which the unambiguous syllabic writing here will categorically not allow.

[15] Thus we should presumably understand line 42 of A 3553, an unprovenanced OB coefficient list [15, text A], as such an inaccurate but arithmetically convenient approximation:

  1 21 IGI.GUB *ṣí-li-ip-tum*   1;21, the coefficient of a diagonal.

[16] These calculations involve some substantial sexagesimal arithmetic, on which see below. And, to save the interested reader more heavy arithmetic, here are some squares:

number	square
1;24 51 09	1;59 59 56 48 19 21
1;24 51 10	1;59 59 59 38 01 40
1;24 51 11	2;00 00 02 27 44 01
1;24 51 12	2;00 00 05 17 26 24

This shows that 1;24 51 10 is the best four-place approximation to $\sqrt{2}$.

tables, but so far nothing like these has been found in the times we have been discussing here.

Both of these alternatives remain conjectural, however, as there is still no direct evidence—by which we mean explicit instructions in the course of a mathematical solution—for the use of more than the first step of the procedure. It is worth finishing with some further remarks about this.

The procedure described here is *mathematically* equivalent to the so-called Heron's method for the extraction of a square root (though it may derive ultimately from Archimedes), found at *Metrica* I 8, where he evaluates an approximation to $\sqrt{720}$:[17]

> Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square root nearest to 720 is 729, having a root 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$. Take half of this; the result is $26\frac{1}{2}\frac{1}{3}$. Therefore the square root of 720 will be very nearly $26\frac{1}{2}\frac{1}{3}$. For $26\frac{1}{2}\frac{1}{3}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $\frac{1}{36}$. If we wish to make the difference less than $\frac{1}{36}$, instead of 729 we shall take the number now found, $720\frac{1}{36}$, and by the same method we shall find an approximation differing by much less than $\frac{1}{36}$. [26, 470–472]

In symbols: if a is an approximation to $\sqrt{N}$, then the average $\frac{1}{2}(a + N/a)$ will be a much better approximation. And if we write our OB procedure as $(a + \frac{1}{2}(N - a^2)/a)$, we immediately see that they are equivalent. But it is *computationally* much more tedious to evaluate $(N - a^2)/a$ than N/a, especially when N is some simple number such as 2.

Take, for example, the case given above of $N = 2$ and $a = 1;25\ 20$. Using "Heron's method," $1/a = 0;42\ 11\ 15$, $2/a = 1;24\ 22\ 30$, and we average in one simple operation to get the next approximation $1;24\ 51\ 15$ in about three lines of work. Now consider our proposed OB procedure. Start by squaring $1;25\ 20$; here we are lucky since all of the intermediate products except for 25×25 are to be found in the standard set of multiplication tables,[18] so five or six lines of careful work yield $2;01\ 21\ 64\ 40$. Then we evaluate $\frac{1}{2} \times 0;01\ 21\ 46\ 40 \times 0;42\ 11\ 15$; here there are eight intermediate nonstandard multiplications to be evaluated before we arrive at the result $\frac{1}{2} \times 0;00\ 57\ 30$, and so the required approximation $1;25\ 20 - 0;00\ 28\ 45 = 1;24\ 51\ 15$.

This may give another possible reason that we only ever find one step of the OB procedure: it is too tedious to contemplate using anything more than a two-sexagesimal-place initial approximation, even when this initial approximation has been chosen or arranged to be regular.[19]

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[17] Heron is describing how to find the area of a triangle with sides 7, 8, and 9 using what we now describe by the formula $\sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$.

[18] This comprised up to 40 multiplication tables of one-, two-, and three-place sexagesimally regular numbers collected together on a single large tablet ("combined tables" in Neugebauer's terminology); individual tables ("single tables") from that standard set might also be written on smaller tablets. The tables, whether single or combined, may take any one of three different formats (with a further four rare and minor variants), but there are no *non*standard multiplication tables known. See, for instance, the 78 single multiplication tables and the 30 combined tables in [19, 19–33; and notes 80 & 90 for references to earlier literature]. Many more have been found since; see most recently [18, 242–245].

[19] After we had completed this article we saw a preprint of [9], which gives another account of Old (and Late) Babylonian square root approximation procedures.

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REFERENCES

1. John P. Britton, A Table of 4th Powers and Related Texts from Seleucid Babylon, *Journal of Cuneiform Studies* **43–45** (1991–1993), 71–78.

2. Evert M. Bruins, Tables of Reciprocals with Irregular Entries, *Centaurus* **17** (1972–1973), 177–188.

3. Evert M. Bruins and Marguerite Rutten, *Textes mathématiques de Suse,* Mémoires de la Mission Archéologique en Iran, vol. 34, Paris: Librairie Orientaliste Paul Geuthner, 1961.

4. Miguel Civil, Sur les ''livres d'écolier'' à l'époque paléo-babylonienne, in *Miscellanea Babylonica: Mélanges offerts à Maurice Birot*, ed. Jean-Marie Durand and Jean-Robert Kupper, Paris: Éditions Recherche sur les Civilisations, 1985, pp. 67–78.

5. David H. Fowler, *Dýnamis, Miṭḥartum*, and Square, *Historia Mathematica* **19** (1992), 418–419.

6. Jöran Friberg, *A Survey of Publications on Sumero-Akkadian Mathematics, Metrology and Related Matters (1854–1982),* Preprint 1982-17, Göteborg: Department of Mathematics, Chalmers University of Technology and University of Göteborg, 1982.

7. Jöran Friberg, Mathematik, in *Reallexikon der Assyriologie*, ed. Dietz Otto Edzard *et al.*, vol. 7, Berlin/Leipzig: de Gruyter, 1928– , pp. 531–585.

8. Jöran Friberg, Five Fragments of Later Babylonian Extended Tables of Reciprocals, in *Cuneiform Texts from the Metropolitan Museum of Art,* ed. Ira Spar, vol. 2, to appear.

9. Jöran Friberg, ''Seed and Reeds Continued,'' Another Metro-Mathematical Topic Text from Late Babylonian Uruk, *Baghdader Mitteilungen,* **28** (1997), 251–365.

10. Cyril J. Gadd, Forms and Colours, *Revue d'assyriologie* **19** (1922), 149–158.

11. Andrew R. George, Ninurta-pāqidāt's Dog-Bite, and Notes on Other Comic Tales, *Iraq* **60** (1993), 63–75.

12. Albrecht Goetze, A Mathematical Compendium from Tell Harmal, *Sumer* **7** (1951), 126–155.

13. Jens Høyrup, *Dýnamis*, the Babylonians, and *Theaetetus* 147c7–148d7, *Historia Mathematica* **17** (1990), 201–222; *Historia Mathematica* **18** (1991), 19. [Also see [5]]

14. Jens Høyrup, Algebra and Naive Geometry: An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, *Altorientalische Forschungen* **17** (1990), 27–69, 262–354.

15. Anne Draffkorn Kilmer, Two New Lists of Key Numbers for Mathematical Operations, *Orientalia* **19** (1960), 273–308.

16. Donald E. Knuth, Ancient Babylonian Algorithms, *Communications of the Association for Computing Machinery* **15** (1972), 671–677; correction: Reciprocal Table of Inakibit-Anu Incorrectly Rendered, *Communications of the Association for Computing Machinery* **19** (1976), 108.

17. Karen Rhea Nemet-Nejat, *Cuneiform Mathematical Texts as a Reflection of Everyday Life in Mesopotamia*, American Oriental Series, vol. 75, New Haven, CT: American Oriental Society, 1993. [Also see [23]]

18. Karen Rhea Nemet-Nejat, Systems for Learning Mathematics in Mesopotamian Scribal Schools, *Journal of Near Eastern Studies* **54** (1995), 241–260.

19. Otto Neugebauer and Abraham Sachs, *Mathematical Cuneiform Texts*, American Oriental Series, vol. 29, New Haven, CT: American Oriental Society and the American Schools of Oriental Research, 1945.

20. Otto Neugebauer, *The Exact Sciences in Antiquity*, 2nd ed., Providence, RI: Brown University Press, 1957; reprint ed., New York: Dover, 1969.

21. Marvin A. Powell, Masse und Gewichte, in *Reallexikon der Assyriologie*, ed. Dietz Otto Edzard *et al.*, vol. 7, Berlin/Leipzig: de Gruyter, 1928– , pp. 457–530.

22. Eleanor Robson, *Old Babylonian Coefficient Lists and the Wider Context of Mathematics in Ancient Mesopotamia, 2100–1600* BC, Oxford: University of Oxford D.Phil. thesis, 1995. [To appear as *Mesopotamian Mathematics, 2100–1600* BC*: Technical Constants in Bureaucracy and Education*, Oxford Editions of Cuneiform Texts, vol. 14, Oxford: Clarendon Press.]

23. Eleanor Robson, Review of [17], *Bibliotheca orientalis* **52** (1995), 424–432.

24. Eleanor Robson, Three Old Babylonian Methods for Dealing with ''Pythagorean'' Triangles, *Journal of Cuneiform Studies*, to appear.

25. H. W. F. Saggs, A Babylonian Geometrical Text, *Revue d'assyriologie* **54** (1960), 131–145.

26. Ivor Thomas [=Bulmer-Thomas], *Selections Illustrating the History of Greek Mathematics*, Loeb Classical Library, vol. 2, Cambridge, MA: Harvard University Press and London: Heinemann, 1941.